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Non-commutative solitons

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Abstract: We find classically stable solitons (instantons) in odd (even) dimensional scalar non-commutative field theories whose scalar potential, $V(\phi)$, has at least two minima. These solutions are bubbles of the false vacuum whose size is set by the scale of noncommutativity. Our construction uses the correspondence between non-commutative fields and operators on a single particle Hilbert space. In the case of non-commutative gauge theories we note that expanding around a simple solution shifts away the kinetic term and results in a purely quartic action with linearly realised gauge symmetries.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry, Field Theories in Lower Dimensions.

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1. Introduction

Quantum field theory on a non-commutative space is of interest for a variety of reasons. It appears to be a self-consistent deformation of the highly constrained structure of local quantum field theory. Non-commutative field theories are nonlocal; unraveling the consequences of the breakdown of locality at short distances may help understanding non-locality in quantum gravity. The discovery of non-commutative quantum field theory in a limit of string theory [1] provides new inroads to the subject.

Perturbative aspects of non-commutative field theories have been analyzed in [2]–[27]. This study has thrown up some evidence for the renormalizability of a class of non-commutative field theories, and has revealed an intriguing mixing of the UV and IR [15] in these theories. In this paper we will construct localized classical solutions in some simple non-commutative field theories. We expect these objects to play a role in the quantum dynamics of the theory.

We first consider a scalar field with a polynomial potential. A scaling argument due to Derrick [28] shows that, in the commutative case, solitonic solutions do not exist in more than $1+1$ dimensions, as the energy of any field configuration can always be lowered by shrinking. Perhaps surprisingly, for sufficiently large noncommutativity parameter θ, we will find classically stable solitons in any theory with a scalar potential with more than one *local* minimum. These solitons are asymptotic to the true vacuum, and reach a second (possibly false) vacuum in their core. They cannot decay simply by shrinking to zero size because sharply peaked field configurations have high energies in non-commutative field theories. These solitons are metastable in the quantum theory, but by adjusting parameters in the scalar potential, their lifetime can be made arbitrarily long while their mass is kept fixed. Solutions are found corresponding to solitons in $2l+1$ dimensions or instantons in $2l$ dimensions for any l.

Our construction of these solutions exploits the connection between non-commutative fields and operators in single particle quantum mechanics. Under this correspondence, the $\star$ product maps onto usual operator multiplication, and the equation of motion translates into algebraic operator equations. The non-commutative scalar action can be rewritten as the trace over operators (which can be regarded as $\infty \times \infty$ matrices). This leads to a connection between non-commutative field theories, and zero-dimensional matrix models.

Next we consider non-commutative $\mathrm{U}(N)$ Yang-Mills theory. When expanded around a simple solution of the equations of motion, the action takes the simple quartic form (up to constants and topological terms)

$$S_{\mathrm{YM}} = \frac{1}{4g_{\mathrm{YM}}^2} \int d^{2l}x \, \delta^{\mu\lambda} \, \delta^{\nu\rho} \, \mathrm{Tr} \left([\Phi_\mu, \Phi_\nu][\Phi_\lambda, \Phi_\rho] \right), \tag{1.1}$$

where Φ_μ are $N \times N$-hermitean matrices and all commutators are constructed from the $\star$ product. Note that the kinetic term has been shifted away! The usual space-time gauge symmetries act linearly as unitary transformations on the fields Φ_μ, and the $\Phi_\mu = 0$ vacuum leaves even local gauge symmetries unbroken. This construction is similar to that of [29], in which the kinetic term of Witten's string field theory action [30] is shifted away. Indeed, our search for such a construction in non-commutative field theory was motivated by the tantalizing analogy, noted in [15], between non-commutative field theories and string field theories. The existence of the formulation (1.1) of non-commutative gauge theories strengthens the analogy. We also reproduce, as an illustration, the U(1) instanton solutions of [31].

Rewriting non-commutative fields as the large-N limit of matrices, (1.1) is closely related to the IKKT matrix theory [32]. Indeed, our construction is essentially equivalent to that presented by Aoki et al. [11] in this context. Related observations are also made in [37, 1, 33, 36].

This paper is organized as follows. In section 2 we describe the action for non-commutative scalar field theory. In section 3 we consider the limit, $\theta \to \infty$, in which the equations simplify considerably. The general solution can be found exactly and is given in terms of quantum mechanical projection operators. In section 4 we show that there are stable solitons in this limit, as long as the potential has at least two local minima. In section 5 we argue that there are stable solitons at large but finite θ which can be constructed perturbatively in θ^{-1}. In section 6 we turn to the non-commutative gauge theory where the purely quartic action is constructed. The U(1)-instanton solution of [31] is also reproduced. In appendix A we give an explicit construction, of the leading $1/\theta$ correction to the simplest stable soliton of the scalar field theory.

2. The non-commutative scalar action

Consider first a non-commutative field theory of a single scalar ϕ in $(2+1)$ dimensions with noncommutativity purely in the spatial directions. The spatial R^2 is parametrized by complex coordinates $z, \bar{z}$. The energy functional

$$E = \frac{1}{g^2} \int d^2z \left(\partial_z \phi \, \partial_{\bar{z}} \phi + V(\phi) \right), \tag{2.1}$$

where $d^2z = dx\, dy$. (We will comment on the generalization to arbitrary dimensions in the appropriate places.) Fields in this non-local action are multiplied using the Moyal star product,

$$(A \star B)(z, \bar{z}) = e^{\frac{\theta}{2}(\partial_z \partial_{\bar{z}'} - \partial_{z'} \partial_{\bar{z}})} A(z, \bar{z}) B(z', \bar{z}') \left.\right|_{z=z'} . \tag{2.2}$$

Note that in the quadratic part of the action, the star product reduces to the usual product.

We seek finite energy (localized) solitons of (2.1). These can also be interpreted as finite action instantons in the two-dimensional euclidean theory. We will, however, refer to the solutions as solitons in the following.

Since no solutions exist in the commutative limit $\theta = 0$ [28], we begin our search in the limit of large noncommutativity, $\theta \to \infty$. It is useful to non-dimensionalize the coordinates $z \to z\sqrt{\theta}$, $\bar{z} \to \bar{z}\sqrt{\theta}$. As a result, the $\star$ product will henceforth have no θ, i.e. it will be given by (2.2) with $\theta = 1$. Written in rescaled coordinates, the dependence on θ in the energy is entirely in front of the potential term:

$$E = \frac{1}{g^2} \int d^2z \left(\frac{1}{2} (\partial \phi)^2 + \theta V(\phi) \right) . \tag{2.3}$$

In the limit $\theta \to \infty$, with V held fixed, the kinetic term in (2.3) is negligible in comparison to $V(\phi)$, at least for field configurations varying over sizes of order one in our new coordinates.

Our considerations apply to generic potentials $V(\phi)$, but we will, for definiteness, mostly discuss those of polynomial form

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \sum_{j=3}^{r} \frac{b_j}{j} \phi^j \,. \tag{2.4}$$

We have, of course, abbreviated

$$\phi^j = \phi \star \phi \star \cdots \star \phi \,.$$

3. Scalar solitons in the $\theta = \infty$ limit

After neglecting the kinetic term, the energy

$$E = \frac{\theta}{g^2} \int d^2 z \, V(\phi) \tag{3.1}$$

is extremised by solving the equation

$$\frac{\partial V}{\partial \phi} = 0 \,. \tag{3.2}$$

For instance, (3.2) is

$$m^2 \phi + b_3 \, \phi \star \phi = 0 \tag{3.3}$$

for a cubic potential and

$$m^2 \phi + b_3 \, \phi \star \phi + b_4 \, \phi \star \phi \star \phi = 0 \tag{3.4}$$

for a quartic potential.

If $V(\phi)$ were the potential in a commutative scalar field theory, the only solutions to (3.2) would be the constant configurations

$$\phi = \lambda_i \,, \tag{3.5}$$

where $\lambda_i \in \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ are the various real extrema of the function $V(x)$.[1] As we shall see below, the derivatives in the definition of the star product allow for more interesting solutions of (3.2).

[1] For $V(\phi)$ as in (2.4), λ_i are the real roots of the equation $m^2 x + \sum_{j=3}^{r} b_j x^{j-1} = 0$.

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3.1 A simple non-trivial solution

A non-trivial solution to (3.2) can easily be constructed. Given a function $\phi_0(x)$ that obeys

$$(\phi_0 \star \phi_0)(x) = \phi_0(x)\,, \tag{3.6}$$

it follows by iteration that $\phi_0^n(x) = \phi_0(x)$,[2] and that $f(a\phi_0(x)) = f(a)\phi_0(x)$ (fields in f are multiplied using the star product). In particular, $\lambda_i\,\phi_0(x)$ solves (3.2) when λ_i is an extremum of $V(x)$. Thus, in order to find a solution of (3.2), it is sufficient to find a function that squares to itself under the star product. We proceed to construct such a function below.

If we take the ordinary product of a smooth function of width Δ with itself, the spatial size of the function shrinks to a fraction of Δ, which is why non-constant functions never square to themselves! The non locality of the star product, however, introduces an additional effect, adding roughly[3] $1/\Delta$ to the width of the product. This makes it possible for a lump of approximately unit size to square to itself under the star product.

Consider a gaussian packet of the form

$$\psi_\Delta(r) = \frac{1}{\pi\Delta^2}\, e^{-r^2/\Delta^2}\,,$$

with radial width Δ (here $r^2 = x^2 + y^2$). The star product of ψ_Δ with itself is easily computed by passing to momentum space,

$$\tilde{\psi}_\Delta(k) = \int e^{ik\cdot x}\,\psi_\Delta(x)\,d^2x = e^{-k^2\Delta^2/4}\,, \tag{3.7}$$

$$\begin{aligned}\left(\tilde{\psi}_\Delta \star \tilde{\psi}_\Delta\right)(p) &= \frac{1}{(2\pi)^2}\int d^2k\,\tilde{\psi}_\Delta(k)\tilde{\psi}_\Delta(p-k)\,e^{\frac{i}{2}\epsilon_{\mu\nu}k^\mu(p-k)^\nu} = \\ &= \frac{1}{2\pi\Delta^2}\,e^{-\frac{p^2}{8}\left(\Delta^2+\frac{1}{\Delta^2}\right)}\,.\end{aligned} \tag{3.8}$$

Therefore

$$(\psi_\Delta \star \psi_\Delta)(x) = \frac{1}{\pi^2\Delta^2(\Delta^2+\frac{1}{\Delta^2})}\exp\left[\frac{-2r^2}{\Delta^2+\frac{1}{\Delta^2}}\right]. \tag{3.9}$$

In particular,[4] when $\Delta^2 = 1$, the gaussian squares to itself (up to a factor of 2π). That is,

$$\phi_0(x) = 2\pi\psi_1(x) = 2\,e^{-r^2} \tag{3.10}$$

solves (3.6) and $\lambda_i\,\phi_0(x)$ solves (3.2).

[2] This equation and its solution has also appeared in earlier work involving the Moyal Product. See [38, 39].

[3] The added width is actually $\approx K$, the typical momentum in the Fourier transform of the function. For a function of size Δ with no oscillations, $K \approx 1/\Delta$. For a function of size Δ with n oscillations, $K \approx n/\Delta$.

[4] We note in passing that in the limit $\Delta \to 0$, (3.9) reduces to $\delta^2(x) \star \delta^2(x) = 1/(2\pi)^2$.

3.2 The general solution

In order to find all solutions of (3.2) we will exploit the connection between Moyal products and quantization. Given a C^∞ function $f(q,p)$ on R^2 (thought of as the phase space of a one-dimensional particle), there is a prescription which uniquely assigns to it an operator $O_f(\hat{q},\hat{p})$, acting on the corresponding single particle quantum mechanical Hilbert space, $\mathcal{H}$. It is convenient for our purposes to choose the Weyl or symmetric ordering prescription

$$O_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int d^2k\, \tilde{f}(k)\, e^{-i(k_q\hat{q}+k_p\hat{p})}, \tag{3.11}$$

where

$$\tilde{f}(k) = \int d^2x\, e^{i(k_q q+k_p p)} f(q,p) \tag{3.12}$$

and

$$[\hat{q},\hat{p}] = i\,. \tag{3.13}$$

With this prescription, it may be verified that

$$\frac{1}{2\pi}\int dp\, dq\, f(q,p) = \mathrm{Tr}_{\mathcal{H}}\, O_f \tag{3.14}$$

and that the Moyal product of functions is isomorphic to ordinary operator multiplication

$$O_f \cdot O_g = O_{f\star g}\,. \tag{3.15}$$

In order to solve any algebraic equation involving the star product, it is thus sufficient to determine all operator solutions to the equation in $\mathcal{H}$. The functions on phase space corresponding to each of these operators may then be read off from (3.11). We will now employ this procedure to find all solutions of (3.2).

As noted above, any solution to (3.6) may be rescaled into a solution of (3.2). Particular solutions of (3.2) may thus be obtained by constructing operators in $\mathcal{H}$ that obey (3.6), i.e. $O_\phi^2 = O_\phi$. This equation is solved by any projection operator in $\mathcal{H}$. $\mathcal{H}$ possesses an infinite number of projection operators, which can be classified by the dimension of the subspace they project onto. Each class contains a large continuous infinity of operators, each of which, upon rescaling, yields a solution to (3.2).

The most general solution to (3.2) hence takes the form

$$O = \sum_j a_j P_j\,, \tag{3.16}$$

where $\{P_j\}$ are mutually orthogonal projection operators onto one-dimensional subspaces, with a_j taking values in the set $\{\lambda_i\}$ of real extrema of $V(x)$.

In order to obtain the functions in space corresponding to the solutions (3.16), it is convenient to choose a particular basis in $\mathcal{H}$. Let $|n\rangle$ represent the energy eigenstates of the one-dimensional harmonic oscillator whose creation and annihilation operators are defined by

$$a = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}\,; \qquad a^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}}\,. \tag{3.17}$$

Note that $a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. Any operator may be written as a linear combination of the basis operators $|m\rangle\langle n|$'s, which, in turn, may be expressed in terms of a and $a^\dagger$ as

$$|m\rangle\langle n| =: \frac{a^{\dagger m}}{\sqrt{m!}}\, e^{-a^\dagger a}\, \frac{a^n}{\sqrt{n!}} :\,, \tag{3.18}$$

where double dots denote normal ordering.

We will first describe operators of the form (3.16) that correspond to radially symmetric functions in space. As $a^\dagger a \approx r^2/2$, operators corresponding to radially symmetric wavefunctions are functions of $a^\dagger a$. From (3.18), the only such operators are linear combinations of the diagonal projection operators $|n\rangle\langle n| = 1/n! : a^{\dagger n} e^{-a^\dagger a} a^n :$. Hence all radially symmetric solutions of (3.2) correspond to operators of the form $O = \sum_n a_n |n\rangle\langle n|$, where the numbers a_n can take any values in the set $\{\lambda_i\}$.

We now translate these operator solutions back to field space. From the Baker-Campbell-Hausdorff formula

$$e^{-i(k_q \hat{q} + k_p \hat{p})} = e^{-i(k_{\bar{z}} a + k_z a^\dagger)} = e^{-k^2/4} : e^{-i(k_{\bar{z}} a + k_z a^\dagger)} :\,, \tag{3.19}$$

where

$$k_z = \frac{k_x + i k_y}{\sqrt{2}}\,, \qquad k_{\bar{z}} = \frac{k_x - i k_y}{\sqrt{2}}\,, \qquad k^2 = 2 k_z\, k_{\bar{z}}\,.$$

Any operator O expressed as a normal ordered function of a and $a^\dagger$, $f_N(a, a^\dagger)$, can be rewritten in Weyl ordered form as follows. By definition,

$$O =: f_N(a, a^\dagger) := \frac{1}{(2\pi)^2} \int d^2k\, \tilde{f}_N(k) : e^{-i(k_{\bar{z}} a + k_z a^\dagger)} : \,. \tag{3.20}$$

Using (3.19) and (3.20) may be rewritten as

$$O = \frac{1}{(2\pi)^2} \int d^2k\, \tilde{f}_N(k) e^{k^2/4} e^{-i(k_{\bar{z}} a + k_z a^\dagger)}\,. \tag{3.21}$$

Thus, the momentum space function $\tilde{f}$ associated with the operator O, according to the rule (3.11) is

$$\tilde{f}(k) = e^{k^2/4} \tilde{f}_N(k)\,. \tag{3.22}$$

For the operator $O_n = |n\rangle\langle n|$ we find, using (3.18) and (3.20), that the corresponding normal ordered function $\tilde{\phi}_N^{(n)}(k) = 2\pi e^{-k^2/2} L_n(k^2/2)$. (3.22) then becomes

$$|n\rangle\langle n| = \frac{1}{(2\pi)} \int d^2k\, e^{-k^2/4} L_n\left(\frac{k^2}{2}\right) e^{-i(k_{\bar{z}} a + k_z a^\dagger)}\,, \tag{3.23}$$

where $L_n(x)$ is the n-th Laguerre polynomial. The field $\phi_n(x,y)$ that corresponds to the operator $O_n = |n\rangle\langle n|$ is, therefore,

$$\phi_n(r^2 = x^2+y^2) = \frac{1}{(2\pi)} \int d^2k \, e^{-k^2/4} L_n\left(\frac{k^2}{2}\right) e^{-ik.x} = 2(-1)^n e^{-r^2} L_n\left(2r^2\right). \quad (3.24)$$

Note that $\phi_0(r^2)$ is precisely the gaussian solution found in section 3.1.

In summary, (3.2) has an infinite number of real radial solutions, given by

$$\sum_{n=0}^{\infty} a_n \, \phi_n(r^2)\,, \quad (3.25)$$

where $\phi_n(r^2)$ is given by (3.24) and each a_n takes values in $\{\lambda_i\}$.

In order to generate all non-radially symmetric solutions to (3.2), we rewrite (3.1) in operator language, using (3.14) as

$$E = \frac{2\pi\theta}{g^2} \operatorname{Tr} V(O_\phi)\,. \quad (3.26)$$

Eq. (3.26) is manifestly invariant under unitary transformations of O_ϕ and so has a U(∞) global symmetry. In other words, if O is a solution to the equation of motion, so is $UOU^\dagger$, where U is any unitary operator acting on $\mathcal{H}$. A general hermitean operator (one that corresponds to a real field ϕ) may be obtained by acting on a diagonal operator (i.e. an operator that corresponds to a radially symmetric field configuration) by an element of the U(∞) symmetry group (since any hermitean operator is unitarily diagonalizable). Thus every solution to (3.2) may be obtained from a radially symmetric solution by means of U(∞) symmetry transformations.

Therefore solutions to (3.2) consist of disjoint infinite-dimensional manifolds labelled by the set of eigenvalues of the corresponding operator. Points on the same manifold can be mapped into each other by U(∞) transformations. Each manifold includes several[5] diagonal operators (radially symmetric solutions). We will have more to say about the moduli space of these solutions in the next section.

As all solutions are related to radially symmetric solutions by a symmetry transformation, we will mostly discuss only radially symmetric solutions.

3.3 UV/IR mixing

$\phi_0(r^2)$, the gaussian solution worked out in subsection 3.1, is a lump of unit size centered at the origin, as shown in figure 1.

[5]Distinct diagonal operators having the same eigenvalues lie on the same manifold, being related by the "Weyl" subgroup of U(∞) that permutes eigenvalues.

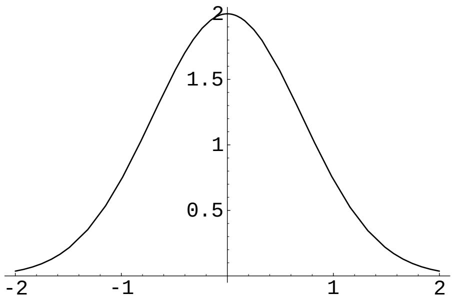

Figure 1: A plot of $\phi_0(r)$ versus r. The solution is a blob centered at the origin.

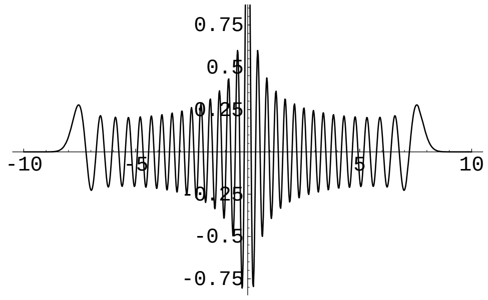

Figure 2: A plot of $\phi_{30}(r)$ versus r.

$\phi_n(r)$, at large n, looks quite different (see figure 2). It is a solution of size $\approx \sqrt{n}$ that undergoes n oscillations[6] in that interval, with oscillation period $\propto 1/\sqrt{n}$. $\phi_n(r^2)$ thus receives significant contributions from momenta up to $\sqrt{n}$ in momentum space. These solutions exemplify the UV/IR mixing pointed out in [15]; oscillations with frequency $\sqrt{n}$ produce an object of size $\sqrt{n}$ (instead of $1/\sqrt{n}$) in a non-commutative theory.

3.4 Generalization to higher dimensions

All considerations of the preceding subsections may easily be generalized to higher dimensions. Consider a scalar field theory in $2l+1$ dimensions with noncommutativity only in the spatial directions. By a choice of axes, the $2l\times 2l$-dimensional noncommutativity matrix Θ may always be brought into block diagonal form. In other words, it is possible to choose spatial coordinates $z_i, \bar{z}_{\bar{j}}$ $(i, \bar{j} = 1, \ldots, l)$, in terms of which the noncommutativity matrix $\Theta_{i\bar{j}} = \theta_i \delta_{i\bar{j}}$, $\Theta_{ij} = \Theta_{\bar{i},\bar{j}} = 0$. As before we consider the limit where θ_i are uniformly taken to ∞ and non-dimensionalize $z_i \rightarrow z_i\sqrt{\theta_i}$. As in the previous subsections, the kinetic term in the action may be dropped in this limit. Solutions to the equations of motion (3.2) are once again in correspondence with operator solutions to the same equations; the operators in question now acting on $\mathcal{H}\times\mathcal{H}\times\cdots\times\mathcal{H}$, l copies of the Hilbert space of the previous subsection. The general solution to (3.2) once again takes the form (3.16) in terms of projection operators on this space. As in the previous subsection, the general solution may be obtained from diagonal solutions via $\mathrm{U}(\infty)$ rotations. Diagonal solutions to (3.2) are given by

$$O = \sum_{\vec{n}} a_{\vec{n}} |\vec{n}\rangle\langle\vec{n}| \longleftrightarrow \sum_{\vec{n}} a_{\vec{n}} \prod_i \phi_{n_i}(|z_i|^2)\,, \tag{3.27}$$

[6] Using asymptotic formulae for Laguerre polynomials we find

$$\phi_n(r) = \begin{cases} 2(-1)^n & r \ll \sqrt{\frac{1}{8n}} \\ \frac{2(-1)^n}{(2\pi^2 r^2)^{1/4}} \cos\left(\sqrt{2n}2r - \frac{\pi}{4}\right) & \sqrt{\frac{1}{8n}} \ll r \ll \sqrt{2n} \\ \frac{2(-2r^2)^n}{n!}\, e^{-r^2} & r \gg \sqrt{2n}. \end{cases}$$

where $\vec{n}$ is shorthand for the set of quantum numbers $\{n_i\}$ for the l-dimensional oscillator and ϕ_{n_i} are defined in (3.24). As in (3.25), the coefficients $a_{\vec{n}}$ take values in $\{\lambda_i\}$. A subset of the solutions (3.27) are actually invariant under SO($2l$) rotations and can be written in terms of associated Laguerre polynomials. These are displayed in subsection A.3 of the appendix.

In summary, in the limit of maximal noncommutativity, the construction of solitons in two spatial dimensions generalizes almost trivially to every even spatial dimension.

4. Stability and moduli space at $\theta = \infty$

In this section we study the stability of the solitons constructed in the previous section. We will also describe the moduli space of stable solitons.

4.1 Stability at $\theta = \infty$

We wish to examine the stability of the radial solution

$$\phi(r^2) = \sum_{n=0}^{\infty} \lambda_{a_n} \phi_n(r^2) \tag{4.1}$$

to small fluctuations. Since any U(∞) rotation does not change the energy of our solution (4.1), it is sufficient to study the stability of (4.1) to radially symmetric fluctuations. These are most conveniently parameterized as deformations of the eigenvalues. The energy for an arbitrary radially symmetric state $\phi(r^2) = \sum_{n=0}^{\infty} c_n \phi_n(r^2)$ is

$$E = \frac{2\pi\theta}{g^2} \sum_{n=0}^{\infty} V(c_n)\,.$$

The solution $c_n = \lambda_{a_n}$ is manifestly an extremum of S, as, by definition, λ_{a_i} are extrema of the function $V(x)$. Clearly (4.1) is a local minimum of the energy (and so a stable solution) if, and only if, λ_{a_n} is a local minimum of $V(x)$ for all $0 \leq n \leq \infty$.

As an example consider the cubic potential of figure 3. with a maximum at $\lambda = -1$. In this case, all λ_{a_n} in (4.1) are either zero or -1. The only stable solution is that for which all $\lambda_{a_n} = 0$, i.e. the vacuum. The solution $-\phi_0(r^2)$, for instance, is unstable, as the energy of this field configuration is decreased by scaling this solution by a constant near unity. This instability shows up as a negative eigenvalue of the quadratic form for fluctuations about $-\phi_0(x)$; the corresponding eigenmode $\delta\phi_0$ is $\propto \phi_0$.

On the other hand the field theory with $V(\phi)$ (say, for a quartic potential) graphed in figure 4 has stable solitons; these are solutions of the form $\phi(r^2) = \sum_{n=0}^{\infty} \lambda_{c_n} \phi_n(r^2)$ with λ_{c_n} taking the values of the minima – 0 or $\lambda \approx -1.4$ for all n. In particular $\lambda\phi_0(r^2)$ is a stable solution, manifestly stable to rescalings. Again,

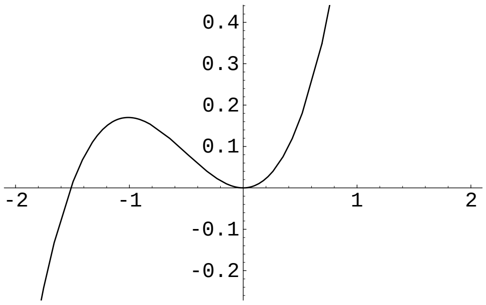

Figure 3: The ϕ^3 theory with an unstable extremum.

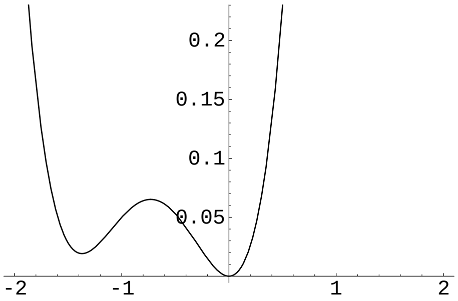

Figure 4: A ϕ^4 potential with two minima.

one may check that the quadratic form for fluctuations about $\lambda\phi_0(r^2)$ is positive. In particular, $\delta\phi \propto \phi_0$ is an eigenmode of this quadratic form with positive eigenvalue.

The stability of $\phi(r^2) = \lambda\phi_0(r^2)$ in the previous example may qualitatively be understood as follows. ϕ_0 is a gaussian of height 2λ. Far away from the origin, $\phi_0(x) = 0$, but near $x = 0$, $\phi_0(x)$ is in the vicinity of the second vacuum. In other words, the static solution corresponds to a bubble of the "false" vacuum. The area of the bubble is of order one (or θ in our original coordinates), the noncommutativity scale. In a commutative theory such a bubble would decay by shrinking to zero size. Noncommutativity prevents the bubble from shrinking to a spatial size smaller than $\sqrt{\theta}$. In order to decay, ϕ_0 actually has to scale to zero — but that process involves going over the hump in the potential and so is classically forbidden.

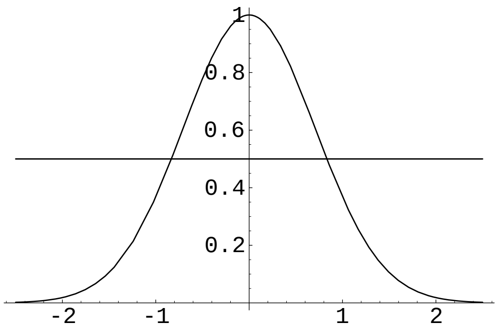

Figure 5: Profile of the gaussian soliton with a false vacuum region (above the horizontal bar) of radius 1.

The energy of this soliton is proportional to the vacuum energy density $V(\lambda)/g^2$ at the "false" vacuum times the volume of the soliton θ. It is remarkable that the energy of the soliton is completely insensitive to the value of the scalar potential at any point except $\phi = \lambda$. Thus the mass of the soliton is unchanged if the height of the barrier in $V(\phi)$ (between $\phi = \lambda$ and $\phi = 0$, see figure 4) is taken to infinity while $V(\lambda)$ is kept fixed. This is true even though $\phi_0(r)$, the solitonic field configuration corresponding to $\lambda|0\rangle\langle 0|$, decreases continuously from $\phi = 2\lambda$ at $r = 0$ to $\phi = 0$ at $r = \infty$!

Consider a 2+1-dimensional scalar theory, noncommutative only in space, at infinite θ. Using the correspondence between functions and operators (matrices) described in the previous section, the non-commutative scalar field theory is equivalent to the matrix quantum mechanics of an $N \times N$-hermitean matrix H, at infinite N, with the usual relativistic kinetic term $\mathrm{Tr}\,(\partial_t H)^2$, and a potential $\mathrm{Tr}\,(V(H))$. The amplitude for an eigenvalue of H to tunnel from λ to 0 is exponentially suppressed by the area under the potential barrier in figure 4, and goes to zero as this area is taken to infinity. Thus the finite mass soliton $\lambda|0\rangle\langle 0|$ is stable, even quantum mechanically, in this limit.

The $\mathrm{U}(\infty)$ symmetry of (3.1) is spontaneously broken by every non-zero solution, $\phi(x)$, of (3.2). As a consequence, every solution has a number of exact zero modes (Goldstone modes) corresponding to small dis-placements about $\phi(x)$ on the manifold of solutions. As $R_{nm} = |n\rangle\langle m| + |m\rangle\langle n|$ and $S_{nm} = i\,(|n\rangle\langle m| - |m\rangle\langle n|)$ are the generators of $\mathrm{U}(\infty)$, these zero modes are given by the non-zero elements of $\delta\phi \propto [R_{nm}, \phi], [S_{nm}, \phi]$.

The $\mathrm{U}(\infty)$ group of symmetry transformations that generates these zero modes is certainly not manifest (at least to the untrained eye) in the energy written in coordinate space in the form (3.1). In addition to the two translations, (3.1) possesses three manifest local symmetries, corresponding to a linear change of the coordinates x, y by an $\mathrm{SL}(2, \mathbb{R})$ matrix. The remaining $\mathrm{U}(\infty)$ transformations act non-locally on $\phi(x, y)$, according to $\phi'(x, y) = (U \star \phi \star U^\dagger)(x, y)$ where $U(x, y)$ is any function that obeys $U \star U^\dagger = 1$ (such functions correspond to $\mathrm{U}(\infty)$ operators under the map (3.11)).

All arguments in this subsection may be applied (after straightforward generalizations) to higher-dimensional solitons.

4.2 Multi solitons

In this subsection we will qualitatively describe a part of the moduli space of stable solitons (at $\theta = \infty$) in the simple case of the potential graphed in figure 4 with a single non-zero minimum at $\phi = \lambda$.

The stable solitons can be characterized by their "level" (number of λ eigenvalues). All stable level one solitons correspond to operators of the form

$$\lambda U \,|0\rangle\, \langle 0\,| U^\dagger\,, \tag{4.2}$$

where U is a unitary operator. As mentioned above, the set of level one solitons span an infinite-dimensional manifold parameterized by $\mathrm{U}(N)/U(N-1)$ (for $N = \infty$).

The soliton looks very different at different points on the manifold. $U = I$ in (4.2) corresponds to the gaussian blob of figure 1. If U happens to be a unitary transformation that maps $|0\rangle$ to $|m\rangle$, for large m, the corresponding wave function is qualitatively similar to that in figure 2. When $U = e^{a^\dagger z - a\bar{z}}$ is the generator of translations, the operator in (4.2), $\lambda|z\rangle\langle z|$, is proportional to the projection operator onto a gaussian centered around $z = \frac{1}{\sqrt{2}}(x + iy)$. (Here $|z\rangle = e^{-|z|^2/2} e^{a^\dagger z}|0\rangle$ is the usual coherent state.) Again, if U corresponds to one of the $\mathrm{SL}(2, \mathbb{R})$ operators, we obtain squeezed states; gaussians elongated in the y direction and shrunk in the x direction. And so on.

Turn now to solitons at arbitrary level n. All such solitons may be obtained by acting on

$$\lambda(\phi_0 + \phi_1 + \cdots + \phi_{n-1})$$

by arbitrary unitary transformations. The manifold of solutions thus generated is parameterized by $\mathrm{U}(N)/\,\mathrm{U}(N-n)$ (and has dimension $d_n \approx 2n\,N$) where $N \to \infty$.

Notice that $d_n \approx n d_1$. This fact has a nice explanation; in a particular limit the manifold of level n solutions reduces to n copies of level 1 solitons very far from each other. This conclusion follows from the observation that the operator that represents n widely separated level one solitons (with centres z_j), for instance

$$M = \lambda \sum_j |z_j\rangle\langle z_j| \tag{4.3}$$

is approximately a level n soliton (and exponentially close to a true level n soliton) when $|z_i - z_j| \to \infty$ for all i, j. We demonstrate this explicitly below for the case $n = 2$.

Using $\langle z| - z\rangle = e^{-2|z|^2}$, it is easy to check that the kets

$$|z_\pm\rangle = \frac{|z\rangle \pm |-z\rangle}{\sqrt{2(1 \pm e^{-2|z|^2})}} \tag{4.4}$$

are orthogonal. From (3.16) we conclude that the projector

$$\begin{aligned} O_z &= \lambda\left(|z_+\rangle\langle z_+| + |z_-\rangle\langle z_-|\right) = \\ &= \lambda \frac{|z\rangle\langle z| + |-z\rangle\langle -z| + e^{-2|z|^2}\left(|z\rangle\langle -z| + |-z\rangle\langle z|\right)}{(1 - e^{-4|z|^2})} \end{aligned} \tag{4.5}$$

corresponds to a level 2 solution. Up to corrections of order $e^{-2|z|^2}$, O_z is equal to $|z\rangle\langle z| + |-z\rangle\langle -z|$, the superposition of field configurations corresponding to two widely separated level one solitons.[7] We conclude that a part of the level n moduli space describes n widely separated level one solitons.

We have, so far, worked in the strict limit $\theta = \infty$. The picture developed in this limit is qualitatively modified at large but finite θ, as we will describe in the next section.

5. Scalar solitons at large but finite θ

We have argued that, under certain conditions on $V(\phi)$, (3.2) has an infinite number of stable solutions. Each solution has an infinite number of exact zero modes, the Goldstone modes of the spontaneously broken $\mathrm{U}(\infty)$ symmetry of (3.1).

At finite θ, the kinetic term in (2.3) explicitly breaks this $\mathrm{U}(\infty)$ symmetry down to the euclidean group in 2 dimensions. Finite θ effects may thus be expected to

[7]It is curious that the kinetic energy of this field configuration is independent of z indicating that there is no force between the two solitons even to next leading order in $1/\theta$.

1. Lift the $\theta = \infty$ manifold of solutions to a discrete set of solutions.

2. Give (positive or negative) masses to the U(∞) Goldstone bosons about these discrete solutions.

In appendix A we will argue that, at large enough θ, corresponding to every radially symmetric solution s of (3.2), there is a radially symmetric saddle point of (2.3), that reduces to s as $\theta \rightarrow \infty$. It is likely that these are the only saddle points of (2.3).

Not all these radially symmetric solutions are stable, however. In fact, it might seem likely that some of the infinite number of zero modes, at $\theta = \infty$, about each solution s, might become tachyonic at finite θ. If this were true, (2.3) would have no classically stable extremum at any finite θ, no matter how large.

We will find that is not the case. In subsection 5.1 below we will argue that any small perturbation of (3.1) must preserve the existence of at least one classically stable level one soliton. In subsection 5.2 we will identify this soliton to be the one near the gaussian $\lambda\phi_0(r^2)$.

5.1 Existence of a stable soliton

For definiteness, through the rest of this section we assume that the potential $V(\phi)$ has the shape shown in figure 4. In particular, it is positive definite. Let the stable extremum of V occur at $\phi = \lambda$ and the unstable extremum at $\phi = \beta$, $(\lambda < \beta < 0)$.

Consider, first, (3.1) i.e. the energy functional in the limit where we neglect the kinetic term. We will show that any path in field space leading from the soliton $\lambda\phi_0(r^2)$ to the vacuum passes through a point whose energy is larger than $\frac{2\pi\theta}{g^2}V(\beta)$. Since the energy of the stable soliton is $\frac{2\pi\theta}{g^2}V(\lambda) < \frac{2\pi\theta}{g^2}V(\beta)$, every path from the soliton to the vacuum must pass over a barrier of height $\mathcal{O}(\theta/g^2)$.

The energy evaluated on an operator A is

$$E = \frac{2\pi\theta}{g^2}\,\mathrm{Tr}(V(A)) = \frac{2\pi\theta}{g^2}\sum_{n=1}^{\infty} V(c_n)\,, \tag{5.1}$$

where c_n are the eigenvalues of A. Since V is positive definite,

$$E \geq \frac{2\pi\theta}{g^2}V(b)\,, \tag{5.2}$$

where b is the smallest eigenvalue of A.

Consider a path in field, or operator space, leading from $\lambda\phi_0$ to the vacuum. At the beginning of this path $b = \lambda$. At its end $b = 0$. Since $\lambda < \beta < 0$, any smooth path with these endpoints must have a point at which $b = \beta$. At that point $E > \frac{2\pi\theta}{g^2}V(\beta)$, as was to be shown.

Now include the kinetic term in (3.1). Barring singular behaviour, this changes the energies of all field configurations by terms of $\mathcal{O}(1/g^2)$. For large enough θ, the arguments of the previous paragraph imply that the field configuration that describe the level one soliton at $\theta = \infty$ cannot decay to the vacuum. Hence there must exist at least one stable soliton near one of the unperturbed level one solutions. In fact, as we will show in the next subsection, there is a stable soliton near the gaussian $\lambda\phi_0(r^2)$. In the appendix we will present an approximate construction of this solution at large but finite θ. A similar argument demonstrates the existence of at least one stable solution at level n.

5.2 Approximate description of the stable soliton

All level one solutions to (3.2) take the form $\lambda U|0\rangle\langle 0|U^\dagger$ where U is a unitary operator. We wish to determine the contribution of the kinetic term to the energy of such an operator.

The kinetic term in (2.1) for an operator A is

$$K = \frac{2\pi}{g^2}\,\mathrm{Tr}[a, A][A, a^\dagger]\,. \tag{5.3}$$

Setting $A = \lambda U|0\rangle\langle 0|U^\dagger$ we find

$$\frac{g^2 K(U)}{2\pi\lambda^2} = 1 + \sum_k 2k|U_{k,0}|^2 - 2\left|\sum_k \sqrt{k+1}U_{k,0}\,U^*_{k+1,0}\right|^2 . \tag{5.4}$$

We expand (5.4) to quadratic order in deviations from $U = I$. Choose $U_i = U_{i,0}$ for $i \geq 1$ as the coordinates for this expansion ($|U_{00}|$ is determined in terms of U_i as U is unitary). To quadratic order in U_i

$$\frac{g^2 K(U)}{2\pi\lambda^2} = 1 + 2\sum_{k=2}^{\infty} k|U_k|^2\,. \tag{5.5}$$

As U_1 and $\bar{U}_1$ do not appear in (5.5), they parameterize flat directions of $K(U)$ (to quadratic order). This was to be expected. Any localized extremum of (2.1) has two exact translational zero modes. Infinitesimally, U_{01} and its complex conjugates act as derivatives on $\phi_0(r^2)$, generating these zero modes. Modulo these zero modes, the fluctuation matrix about $U = 1$ is positive definite.

While $K(U)$ has several critical points other than $U = I$, it has no further local minima. For example, $U = U^{(m)}$, the unitary transformation that rotates $|0\rangle\langle 0|$ to $|m\rangle\langle m|$, is an unstable critical point of $K(U)$ for all m. In fact $U = U^{(m)}$ is unstable to decay into $U = I$. This may be demonstrated by considering the path in field space $|\alpha\rangle\langle\alpha|$ where $|\alpha\rangle = \cos\alpha|0\rangle + \sin\alpha|m\rangle$. (5.3) evaluated on such a path is equal to $1 + 2m\sin^2\alpha$ (for $m > 1$; $1 + 2\sin^4\alpha$ for $m = 1$) indicating that the state $|m\rangle\langle m|$ can decay to $|0\rangle\langle 0|$.

We will now argue that, at large enough θ, the finite θ saddle point $\phi(x, y)$ of (2.3) that reduces to $\lambda|0\rangle\langle 0|$ as $\theta\to\infty$ is classically stable.

Consider the mass matrix for fluctuations about $\phi(x,y)$. Since any operator may be written as $UDU^\dagger$ where D is diagonal and U unitary, small fluctuations may be decomposed into radial (fluctuations of D) and angular ones (fluctuations of U). The mass matrix for purely radial fluctuations is $\mathcal{O}(\theta)$ to leading order, and has been shown to be positive definite in subsection 3.4. The mass matrix for purely angular fluctuations is $\mathcal{O}(1)$ to leading order, and has been shown to be positive definite, modulo the two-zero modes. Since angular modes completely disappear from the potential, mixing between radial and angular fluctuations occurs only through the kinetic term, and are also $\mathcal{O}(1)$. These cross terms result in corrections to the eigenvalues of the mass matrix only at $\mathcal{O}(1/\theta)$. Hence, to leading order in $1/\theta$, the mass matrix is positive. The two-zero modes of the angular mass matrix cannot be driven negative by $1/\theta$ corrections as they are exact.

A similar argument demonstrates the instability of all other radially symmetric level one solitons (those that reduce to $\lambda|n\rangle\langle n|$ at $\theta=\infty$) at large-enough θ.

The considerations of this subsection may easily be generalized to solitons in $2l$ spatial dimensions, using the higher-dimensional analogue of (5.4):

$$\frac{g^2K(U)}{(2\pi)^l\lambda^2} = 1 + 2\sum_{j,\vec{k}} k_j |U_{\vec{k},\vec{0}}|^2 - 2\sum_j \left| \sum_{\vec{k}} \sqrt{k_j+1}\, U_{\vec{k},\vec{0}}\, U^*_{\vec{k}+\vec{i},\vec{0}} \right|^2 \tag{5.6}$$

and (5.5)

$$\frac{g^2K(U)}{(2\pi)^l\lambda^2} = 1 + 2\sum_{\vec{k}} \left(\sum_{j=1}^{l} k_j - \sum_{j=1}^{l} \delta_{\vec{k},\vec{i}} \right) |U_{\vec{k},\vec{0}}|^2 \,. \tag{5.7}$$

We use the notation of (3.27); $\vec{k}$ is an l-dimensional vector, j runs from 1 to l and $\vec{i}$ is the basis unit vector in the i-th direction; in components $i_n = \delta_{i,n}$. Notice that $K(U)$ in (5.7) is independent of $U_{\vec{i},0}$ for all i, a consequence of the exact translational invariance in all $2l$ spatial directions.

6. Non-commutative Yang-Mills

6.1 Quartic action for the U(1) theory in two dimensions

Consider the action

$$S = \frac{1}{4\,g^2_{\rm YM}} \int d^2z \left[\bar\Phi,\Phi\right]\left[\bar\Phi,\Phi\right], \tag{6.1}$$

where Φ is a complex field and

$$\left[\Phi,\bar\Phi\right] \equiv \Phi\star\bar\Phi - \bar\Phi\star\Phi \,. \tag{6.2}$$

The equation of motion following from (6.1) is

$$\left[\bar\Phi,\left[\Phi,\bar\Phi\right]\right] = 0 \,. \tag{6.3}$$

Φ can also be viewed as a quantum mechanical operator and $\bar{\Phi}$ as it's hermitean conjugate. The commutators in (6.1)–(6.3) are then ordinary operator commutators, and the integral is the trace over the Hilbert space. In the operator representation a simple solution of the equation of motion (6.3) is

$$\Phi = a\,, \qquad \bar{\Phi} = a^{\dagger}\,. \tag{6.4}$$

Let us expand around this solution by defining

$$\Phi = a + i\,A_{\bar{z}}\,, \qquad \bar{\Phi} = a^{\dagger} - i\,A_z\,. \tag{6.5}$$

One then finds, translating back to functions (with $\sqrt{2}\,z = q + ip$, $[a,\ ] = \partial_{\bar{z}}$ and $[a^{\dagger},\ ] = -\partial_z$), that

$$\left[\Phi, \bar{\Phi}\right] = 1 + i\,\partial_z A_{\bar{z}} - i\,\partial_{\bar{z}} A_z - [A_z, A_{\bar{z}}] = 1 + i\,F_{z\bar{z}}\,. \tag{6.6}$$

The operator representation of (6.1) has the manifest $\mathrm{U}(N=\infty)$ symmetry under which $\Phi \to \Phi' = U^{\dagger}\Phi U$ just as in the scalar field theory. Infinitesimally,

$$\delta\Phi = i\,[\Phi, \Lambda]\,, \tag{6.7}$$

where $U = \exp i\,\Lambda$. When gauged, this is just the usual $\mathrm{U}(1)$ gauge symmetry of the non-commutative theory,

$$\delta A = d\,\Lambda + i\,[A, \Lambda]\,.$$

The equation of motion (6.3) is

$$D_{\bar{z}} F_{z\bar{z}} = 0\,. \tag{6.8}$$

The action (6.1) is then

$$S = -\frac{1}{4\,g_{\mathrm{YM}}^2} \int d^2 z (F_{z\bar{z}} - i)^2\,, \tag{6.9}$$

the standard two-dimensional non-commutative $\mathrm{U}(1)$ Yang-Mills action up to constants and topological terms.

6.2 The $\mathrm{U}(N)$-theory in $2l$ dimensions

(6.1) can be generalized to

$$S = \frac{1}{4 g_{\mathrm{YM}}^2} \int d^{2l} x \delta^{\mu\lambda} \delta^{\nu\rho}\, \mathrm{Tr}\left([\Phi_\mu, \Phi_\nu][\Phi_\lambda, \Phi_\rho]\right)\,, \tag{6.10}$$

where $\mu, \nu = 1, \ldots, 2l$ and Φ_μ are real $N \times N$ matrices. Though we have restricted ourselves to a flat euclidean metric, one can generalise the argument below to the Minkowski metric as well.

The equation of motion is

$$\delta^{\mu\nu}\left[\Phi_\mu,\left[\Phi_\nu,\Phi_\lambda\right]\right]=0\,. \tag{6.11}$$

We choose complex coordinates such that $\Theta_{a\bar{b}}=i\,\delta_{a\bar{b}}$, with $a,b=1,\ldots,l$. (6.11) has the solution

$$\Phi_b=a_b\,,\qquad \Phi_{\bar{b}}=a^\dagger_{\bar{b}}\,, \tag{6.12}$$

where $[a_b,a^\dagger_{\bar{c}}]=\delta_{b\bar{c}}$. Expanding around this solution with

$$\Phi_b=a_b+iA_{\bar{b}} \tag{6.13}$$

one finds

$$S=-\frac{1}{4\,g^2_{\mathrm{YM}}}\int d^{2l}z\left(F_{a\bar{b}}-\Theta^{-1}_{a\bar{b}}\right)^2. \tag{6.14}$$

As before the manifest $\mathrm{U}(\infty)\otimes\mathrm{U}(N)$ symmetry corresponds to the non-commutative $\mathrm{U}(N)$-gauge symmetry.

6.3 The U(1) instanton

The four-dimensional non-commutative gauge theory has instanton solutions which are deformed versions of the usual non-abelian instantons. In particular, the U(1)-non-commutative theory also has non-singular finite action saddle points [31]. We exhibit the operators Φ_a corresponding to the simplest such U(1) instanton.

The operators Φ_a corresponding to an anti-self dual field strength $\delta^{a\bar{b}}F_{a\bar{b}}=0$ $(a,\bar{b}=1,2)$, obey

$$[\Phi_b,\Phi_c]=0\,,\qquad \delta^{a\bar{b}}[\Phi_a,\Phi_{\bar{b}}]=2\,. \tag{6.15}$$

In four dimensions, the operators $\Phi_a(a=1,2)$ live in a Hilbert space generated by the creation and annihilation operators of a two-dimensional harmonic oscillator (see subsection 3.3). Rather than work in the conventional number basis $|n_1,n_2\rangle$, it is convenient to work in Schwinger's angular momentum basis,

$$|j,m\rangle\equiv\frac{(a^\dagger_1)^{j+m}}{\sqrt{(j+m)!}}\frac{(a^\dagger_2)^{j-m}}{\sqrt{(j-m)!}}|0,0\rangle\,, \tag{6.16}$$

with $0\le j<\infty, |m|\le j$. The operators

$$J_+=a^\dagger_1a_2\,,\qquad J_-=a^\dagger_2\,a_1\,,\qquad J_z=\frac{1}{2}\left(a^\dagger_1a_1-a^\dagger_2\,a_2\right) \tag{6.17}$$

obey the usual angular momentum algebra.

We will find a solution to (6.15) of the form

$$\Phi_b=a_b\sum_{j,m}(1+c_j)|j,m\rangle\langle j,m|=a_b+a_b\sum_{j,m}c_j|j,m\rangle\langle j,m|\,,\qquad \Phi_{\bar{b}}=a^\dagger_{\bar{b}} \tag{6.18}$$

and put it into hermitean form via a complexified gauge transformation W.

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The ansatz (6.18) satisfies the holomorphic part of (6.15) for any c_j. For a real c_j, the only condition comes from the equation $F_{1\bar{1}} = -F_{2\bar{2}}$. Using

$$
\begin{aligned}
a^\dagger_{1,2}\,|j,m\rangle &= \sqrt{j \pm m + 1}\left| j + \frac{1}{2}, m \pm \frac{1}{2}\right\rangle ;\\
a_{1,2}\,|j,m\rangle &= \sqrt{j \pm m}\left| j - \frac{1}{2}, m \mp \frac{1}{2}\right\rangle ,
\end{aligned}
\tag{6.19}
$$

yields the equation $jc_j = (j+1)c_{j+\frac{1}{2}}$. Which has the solution

$$
c_j = \frac{c}{j(2j+1)}\,, \quad (j > 0)\,. \tag{6.20}
$$

The complexified gauge transformation

$$
W = W^\dagger = \sum_{j,m} \sqrt{\frac{j}{j+1}}\,|j,m\rangle \tag{6.21}
$$

puts the solution (6.18) into hermitean form for $c = -1$. The field strength then takes the compact form

$$
[\Phi_{\bar{b}}, \Phi_c] = -\delta_{\bar{b}c} - iF_{\bar{b}c} = -\delta_{\bar{b}c} - (\vec{J}\cdot\vec{\sigma})_{\bar{b}c} \sum_{j,m} \frac{1}{j(j+1)(2j+1)}\,|j,m\rangle\langle j,m|\,. \tag{6.22}
$$

Here $\vec{J}$ are the angular momentum generators defined in (6.17) and $\vec{\sigma}$, the usual Pauli matrices. This solution is exactly the same as the simplest charge one U(1) instanton in [31]. It may be checked that $\frac{1}{2}\,\mathrm{Tr}\,F^2_{a\bar{b}} = 1$.

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A. Solutions at finite θ

In this appendix we will examine radially symmetric saddle points of (2.3) at finite θ. In subsection A.1 we study the equation of motion resulting from (2.3) at finite θ, and examine the existence of radially symmetric solutions to these equations. In subsection A.2 we concentrate on a particular solution; the one that reduces to the stable soliton $\lambda|0\rangle\langle 0|$ as θ is taken to infinity. We present an approximate construction of this soliton at large θ. In subsection A.3 we briefly comment on the generalization of these results to solitons in higher dimensions.

A.1 The perturbation expansion and a recursion relation

The full equation of motion derived from (2.1) may be written in momentum space as

$$\tilde{\phi}(k^2) + \sum_{j=3}^{r} \frac{b_j}{m^2} \tilde{\phi}^{j-1}(k^2) = \frac{-k^2}{m^2\theta}\, \tilde{\phi}\,(k^2) \tag{A.1}$$

While the l.h.s. of (A.1) is independent of θ, the r.h.s. is of order $1/\theta$, and so is a small parameter at large θ. For notational convenience, we set $b_j/m^2 = d_j$ and $1/m^2\theta = \epsilon$.

Let

$$\sum_{n=0}^{\infty} c_n\, \tilde{\phi}_n\, \left(k^2\right) \tag{A.2}$$

be a solution to (A.1). Substituting (A.2) into (A.1), using the recurrence relation for Laguerre polynomials, and equating coefficients of $\tilde{\phi}_n(k^2)$, we arrive at the difference equations

$$c_n + \sum_{j=3}^{r} d_j c_n^{j-1} = 2\,\epsilon \Big[n c_{n-1} - (2n+1) c_n + (n+1) c_{n+1} \Big]\,. \tag{A.3}$$

We are interested in finite energy solutions to (3.2), i.e. solutions to (A.3) for which

$$\sum_{n} V(c_n) < \infty\,. \tag{A.4}$$

Since $V(0) = 0$, (A.4) will be satisfied if the c_ns approach zero sufficiently fast as n approaches infinity. For such a solution, all non-linear terms in (A.3) may be neglected at large-enough n. At sufficiently large n, n may also be replaced by a continuous variable u, and (A.3) turns into the second-order differential equation

$$c(u) = 2\,\epsilon\, u \frac{d^2 c(u)}{du^2}\,. \tag{A.5}$$

(A.5) is the Schroedinger equation for a zero energy state of a particle in a $1/u$ potential. $\sqrt{\epsilon}$ plays the role of Planck's constant, and at small ϵ (A.5) is easily solved in the WKB approximation, yielding

$$c(u) = A_{-} u^{1/4} e^{-\sqrt{2u/\epsilon}} + A_{+} u^{1/4} e^{+\sqrt{2u/\epsilon}}\,, \tag{A.6}$$

where $A_{\pm}$ are arbitrary constants. In order that c_n tend to zero at large n, $A_{+} = 0$. Thus, for large[8] n,

$$c_n \approx A n^{1/4} e^{-\sqrt{2n/\epsilon}}\,. \tag{A.7}$$

[8](A.6) is a good approximation when $|c_n| \ll 1$ (so that dropping non-linear terms in (A.3) is justified) and $\frac{c_n - c_{n-1}}{c_n} \ll 1$, i.e. $n\epsilon \gg 1$ (so that the transition from (A.3) to (A.5) is justified).

Eq. (A.7) has an undetermined parameter A, the scale of the solution at large n. As (A.3) is a non-linear equation, A is not an arbitrary parameter, but is determined to be one of a discrete set of values. Given c_p and c_{p+1}, the $(p+1)$ equations (A.3) with $n = 0, \ldots, p$ overdetermine the p unknowns c_n for $n < p$. The extra equation constrains the scale A, as we will see in the next subsection.

A.2 The gaussian soliton corrected

In this section we present an approximate construction of the stable soliton that reduces to the gaussian at infinite θ. Our construction approximates the true solution to arbitrary accuracy at small enough ϵ.

We wish to find a solution of (A.3) such that

$$\lim_{\epsilon \to 0} c_0 = \lambda \tag{A.8}$$

and

$$\lim_{\epsilon \to 0} c_m = 0 \tag{A.9}$$

uniformly in m, for $m \geq 1$. (A.9) ensures that, on such a solution, (A.3) for $n \geq 1$ reduces to

$$c_n = 2\,\epsilon \Big[n\, c_{n-1} - (2n+1) c_n + (n+1) c_{n+1} \Big] \tag{A.10}$$

for small enough ϵ. It is easy to find an explicit solution to (A.10) that obeys (A.8) and (A.9). Consider a function $\phi(x, y)$ that obeys the differential equation

$$\left(-\epsilon \partial^2 + 1 \right) \phi = b\, \phi_0 \,. \tag{A.11}$$

Expanding ϕ in the form

$$\phi = \sum_{n=0}^{\infty} c_n \phi_n \tag{A.12}$$

and imitating the manipulations of appendix A.1, we find that $c_n s$ obey (A.10) for $n \geq 1$, but obey

$$c_0 = 2\epsilon \left[c_1 - c_0 \right] + b \tag{A.13}$$

instead of (A.3) (with $n = 0$). This relation will fix the free parameter b.

(A.11) is easily solved in momentum space

$$\tilde{\phi}(k) = b\, \frac{\tilde{\phi}_0(k)}{1 + 2\, \epsilon\, k^2} \,. \tag{A.14}$$

Using the explicit forms for $\tilde{\phi}_n(k)$ and orthogonality of the Laguerre polynomials we find

$$c_n = b \int_0^{\infty} dx\, \frac{e^{-x} L_n(x)}{1 + 2\, \epsilon\, x} \,. \tag{A.15}$$

In particular

$$c_0 = b \int dx \, \frac{e^{-x}}{1+2\epsilon x} = b\, F(\epsilon) \quad \text{where} \quad F(\epsilon) = 1 - \epsilon + \mathcal{O}\left(\epsilon^2\right). \tag{A.16}$$

Using (A.16) we conclude that (A.13) and (A.3) (at $n=0$) are identical on $\{c_n\}$ if b is chosen such that

$$b\, F(\epsilon) + \sum_{j=3}^{r} d_j \left(bF(\epsilon)\right)^{j-1} = b\left(F(\epsilon) - 1\right). \tag{A.17}$$

We wish to find a solution to (A.17) that obeys (A.8), i.e. (from (A.16)) one for which $\lim_{\epsilon\to 0} b = \lambda$. As $\lambda + \sum_{j=3}^{r} d_j \lambda^{j-1} = 0$, such a solution exists, and takes the form

$$b\left(\epsilon\right) = \lambda\left(1 + K\epsilon + \mathcal{O}\left(\epsilon^2\right)\right) \tag{A.18}$$

at small ϵ where K is a number that may easily be determined.

In summary, $\{c_n\}$ given by (A.15) with b given by (A.17), (A.18), solve (A.11) for $n \geq 1$ and (A.3) (with $n=0$). $\{c_n\}$ therefore also approximately satisfy the true difference equations (A.3) for all n as long as $|c_n| \ll 1$ for all $n \geq 1$. But it is easy to verify that all $|c_n|$ for all $n \geq 1$ are arbitrarily small at small enough ϵ. Using the completeness of the Laguerre polynomials,

$$\sum_{n=0}^{\infty} c_n^2 = b^2 \int_0^{\infty} \frac{e^{-x}}{(1+2\,\epsilon\, x)^2} < b^2\,. \tag{A.19}$$

But

$$c_0 = b \int_0^{\infty} dx \frac{e^{-x}}{1+2\epsilon x} > b \int_0^{\infty} dx\, e^{-x}(1-2\epsilon x) = b\left(1-2\epsilon\right). \tag{A.20}$$

Combining (A.19) and (A.20)

$$\sum_{n=1}^{\infty} c_n^2 < 4\,\epsilon\, b^2 \tag{A.21}$$

establishing (A.9) uniformly in n on our solution. Thus $\{c_n\}$ provides an approximate solution to the full non-linear difference equations (A.3) for all n at small enough ϵ. Furthermore, from (A.19), this solution has finite energy.

As $\{c_n\}$ obey the linearized recursion relation (A.11) and are small at small ϵ, we can conclude, from the previous subsection, that d_n takes the form (A.7) for $n\,\epsilon \gg 1$. In order to estimate the behaviour of $c_n(\epsilon)$ for $n \ll 1/\epsilon$ we formally expand the denominator in (A.15) in a power series in ϵx and integrate term by term, arriving at the asymptotic expansion

$$c_n = \sum_{m=n}^{\infty} (-1)^{m+n} (2\epsilon)^m \frac{m!^2}{n!(m-n)!}\,. \tag{A.22}$$

This expansion is useful only when the first few terms in the series in (A.22) are successfully smaller, i.e. for $n\epsilon \ll 1$.

A.3 Generalization to higher dimensions

In this subsection we will outline the generalization of the arguments of subsection A.1 and the construction of subsection A.2, for the case of the maximally isotropic noncommutativity in $2l$ dimensions, i.e. a theory with noncommutativity matrix Θ, all of whose eigenvalues are $\pm i\theta$. It is likely that these arguments can be further extended to generic Θ.

We first note that a subset of the diagonal $\theta = \infty$ solutions (3.27) are (in non-dimensionalized coordinates) invariant under SO$(2l)$ rotations. These solutions take the form

$$\sum_{\vec{n}} c_J \frac{1}{\sqrt{D_J}} \delta_{(J, \sum_i n_i)} |\vec{n}\rangle\langle\vec{n}| \longleftrightarrow \frac{1}{\sqrt{D_J}} \sum_J c_J \, \phi_J^{(l)} \left(r^2\right). \tag{A.23}$$

Here

$$\phi_J^{(l)} \left(r^2 = \sum_i |z_i|^2\right) = 2^l (-1)^J L_J^{(l-1)} \left(r^2\right), \tag{A.24}$$

where $L_J^{(l-1)}(r^2)$ is an associated Laguerre polynomial. (A.24) is obtained from (3.27) by repeated use of the identity

$$\sum_{m=0}^{n} L_{n-m}^{\alpha}(x) L_m^{\beta}(y) = L_n^{\alpha+\beta+1}(x+y).$$

$D_J = \binom{J+l-1}{J}$ is a convenient normalization factor.

When the noncommutativity matrix is maximally isotropic, the kinetic term in (2.3) is invariant under SO$(2l)$ rotations of rescaled coordinates. Thus the corrections to an SO$(2l)$-invariant $\theta = \infty$ solution, of the form (A.23), are also SO$(2l)$ invariant.

Restricting to SO$(2l)$-invariant functions, the arguments of subsection A.1 are easily generalized. Any SO$(2l)$ invariant function takes the form

$$\tilde{\phi}\left(k^2\right) = \sum_{n=0}^{\infty} c_J \tilde{\phi}_J\left(k^2\right); \qquad \tilde{\phi}_J\left(k^2\right) = \frac{1}{\sqrt{D_J}} (2\pi)^l L_J^{(l-1)} \left(\frac{k^2}{2}\right) e^{-k^2/4}. \tag{A.25}$$

The equation of motion implies that c_J obey the following generalization of (A.3):

$$c_J + \sum_{j=3}^{r} d_j c_J^{j-1} = 2\,\epsilon \Big[(J+l-1) c_{J-1} - (2J+l) c_J + (J+1) c_{J+1}\Big]. \tag{A.26}$$

For large J (A.26) and (A.3) are identical, hence all conclusions of subsection A.1 carry over to this case.

The perturbative construction of the solution that reduces to the SO$(2l)$ invariant gaussian proceeds as in subsection A.2 yielding the approximate result (good for small ϵ)

$$d_J = \frac{b}{\sqrt{D_J}\,\Gamma(l)} \int_0^{\infty} dx\, \frac{x^{l-1} e^{-x} L_J^{(l-1)}(x)}{1 + 2\,\epsilon\, x}. \tag{A.27}$$

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